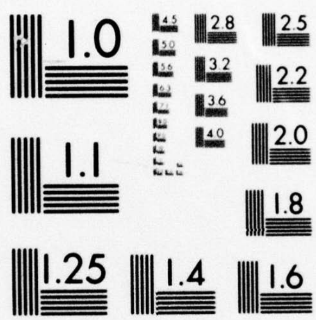
AD-A050 197 MISSOURI UNIV-COLUMBIA DEPT OF STATISTICS F/G 14/4
TWO SIDED CONFIDENCE INTERVALS FOR AN EXPONENTIAL PARAMETER FRO--ETC(U)
JAN 78 K B FAIRBANKS N00014-76-C-0789
UNCLASSIFIED TR-73 NL

| OF |

AD
A050197

END
DATE
FILMED
3-78
DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD A050197

(12)

University of Missouri-Columbia

Two Sided Confidence Intervals
for an Exponential Parameter

by

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
73

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER
9

4. TITLE (and Subtitle)
6 Two Sided Confidence Intervals for an Exponential Parameter from a Life Test with Type I and Type II Censoring.

5. TYPE OF REPORT & PERIOD COVERED
Technical Report.

6. PERFORMING ORG. REPORT NUMBER
14 TR-73

7. AUTHOR(s)
10 Kenneth B. Fairbanks

8. CONTRACT OR GRANT NUMBER(s)
15 N00014-76-C-0789

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Department of Statistics
University of Missouri-Columbia
Columbia, Missouri 65201

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
NR 042-353

11. CONTROLLING OFFICE NAME AND ADDRESS
Office of Naval Research
Department of the Navy
Arlington, Virginia 22217

12. REPORT DATE
11 Jan[redacted] [redacted]78

13. NUMBER OF PAGES
32
12 36p.

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Confidence interval, exponential parameter, life test, type I and type II censoring

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
A life test on items assumed to have an exponential lifetime is often designed with type I censoring, type II censoring or both. In type I censoring n items are placed on test and observed for a fixed time t^*, while in type II censoring the test terminates with the r^{th} failure, where r is a preassigned integer. If these schemes are combined, the test terminates at $\min(t^*, \tau_r)$ where τ_r is the time of the r^{th} failure. Procedures → next page

tau sub r

(OVER)

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

402 660

(ABSTRACT, Continued)

for estimating the exponential parameter from this combined scheme were first considered by Epstein. Epstein established one sided confidence intervals for this parameter. This report reviews Epstein's work and establishes two sided confidence intervals. The confidence intervals are expressed in terms of time on test and chi-square percentiles with the degrees of freedom depending on the number of observed failures. An expression for the expected length of the confidence intervals is also derived.

Two Sided Confidence Intervals for an Exponential Parameter from a Life Test with Type I and Type II Censoring

by

Kenneth B. Fairbanks
University of Missouri - Columbia

Technical Report No. 73

January 1978

Prepared under contract N00014-76-6-0789
(NR-042-353)
Office of Naval Research
Richard W. Madsen, Project Director

University of Missouri
Department of Statistics
Columbia, Missouri 65201

I. Introduction

Reliability acceptance tests are conducted under a variety of test schemes both in the military and in industry. Various combinations of time and failure censoring are commonly utilized. In time censoring, n items are placed on test and observed for a pre-chosen time t^*. In failure censoring, n items are observed until the occurrence of the r^{th} failure. These procedures are sometimes denoted type I and type II censoring respectively. These two schemes may be combined so that the test terminates at $\min(t^*, \tau_r)$, where τ_r is the time of the r^{th} failure in the sample. Sequential tests are also used where acceptance, rejection, and continuation regions are defined in the time-failure plane according to the theory of Wald [10]. The sequential test may be further modified by time censoring, failure censoring, or both. If a sequential test is censored at r_0 failures <u>and</u> at time T_0, the resulting test is known as a truncated sequential test scheme. Further variations are possible by monitoring total accumulated time on test rather than actual elapsed time in any of the above schemes.

A reliability test is concerned with a parameter θ in the underlying distribution of lifetimes. In addition to an accept or reject decision, it might be of interest to form an interval estimate of θ at the time of decision.

The objective of this report is precisely that. If the reliability test terminates at time $t.$, we wish to form a $100(1-\alpha)$ percent confidence interval for θ at that time.

In this report we restrict our work to an underlying distribution of lifetimes which is exponential with parameter θ . This report was motivated by a communication from the Office of Naval Research concerning proposed revisions in MIL-STD-781. This standard gives detailed reliability acceptance procedures for military use, employing schemes which combine types I and II censoring as well as truncated sequential tests. The underlying distribution is assumed to be exponential with parameter θ , which is the mean time between failures (MTBF). The communication indicated a desire to form confidence intervals for θ directly from the testing procedures in MIL-STD-781.

II. Literature Review and Preliminaries

Much of the existing theory on reliability testing as well as estimation procedures, in the exponential case was developed bv Benjamin Epstein and Milton Sobel in a series of papers and technical reports. Epstein eventually unified these papers in a monograph [4] in 1960.

In what follows it is assumed that all test items are drawn at random and independently from a population where the lifetime, τ , of an item is exponentially distributed with density

$$f(\tau;\theta) = \begin{cases} \frac{1}{\theta} e^{-\tau/\theta}, & \tau \geq 0, \ \theta > 0 \\ 0, & \text{elsewhere.} \end{cases} \tag{1}$$

Hence θ is the expected lifetime of any item on test. Suppose n items are drawn at random from distribution (1) and placed on

life test. Define τ_i to be the i^{th} observed failure time. If $r \le n$ is preassigned for type II censoring, we have $\tau_1 \le \tau_2 \le \ldots \le \tau_r$. Failed items may not be replaced (sampling without replacement) or they may be immediately replaced by a new item from the same distribution (sampling with replacement). Under these schemes the following results are well established [4]:

i) In the nonreplacement case the maximum likelihood estimator of θ is $\hat{\theta} = T_r/r$, where

$$T_r = \sum_{i=1}^{r} \tau_i + (n-r)\tau_r = \sum_{i=1}^{r} (n-i+1)(\tau_i - \tau_{i-1}) = \sum_{i=1}^{r} u_i \,.$$

Here $\tau_0 \equiv 0$ and $u_i \equiv (n-i+1)(\tau_i - \tau_{i-1})$. T_r is in fact the total accumulated time on test.

ii) The u_i, $i = 1,\ldots,r$, are independent and identically distributed with common probability density function (1).

iii) $2u_i/\theta$ is distributed as a chi-square random variable with 2 degrees of freedom. Consequently $2T_r/\theta$ is distributed as chi-square with 2r degrees of freedom (χ^2_{2r}).

iv) In the replacement case the maximum likelihood estimator of θ is $\hat{\theta} = T_r/r$, where now $T_r = n\tau_r$.

v) Also in the replacement case $2r\hat{\theta}/\theta = 2T_r/\theta \sim \chi^2_{2r}$.

From these results we can obtain one and two sided confidence intervals for θ, with or without replacement. From (iii) and (v) we find that a $100(1-\alpha)$ percent one sided confidence interval for θ is

$$\left(\frac{2T_r}{\chi^2_{\alpha,2r}}, \infty\right) \tag{2}$$

where $\chi^2_{\alpha,2r}$ is such that $P(\chi^2_{2r} > \chi^2_{\alpha,2r}) = \alpha$. The corresponding two-sided interval is

$$\left(\frac{2T_r}{\chi^2_{\frac{\alpha}{2},2r}}, \frac{2T_r}{\chi^2_{1-\frac{\alpha}{2},2r}}\right) \tag{3}$$

In type I censoring, where testing continues until a preassigned time t^* , confidence intervals are not as easily found. In this case, with n items on test with replacement, we have a Poisson process with parameter $n(\frac{1}{\theta})$. Hence, if k is the number of failures observed in $(0,t^*)$,

$$P(k \le r|\theta) = \sum_{k=0}^{r} \frac{e^{-\frac{nt^*}{\theta}}(\frac{nt^*}{\theta})^k}{k!} .$$

By a well known relationship between the Poisson and chi-square distributions we can write

$$P(k \le r|\theta) = P(\chi^2_{2r+2} > 2nt^*/\theta) .$$

Clearly, if $\theta \le 2nt^*/\chi^2_{\alpha,2r+2}$, then $P(k \le r) \le \alpha$. So, if we do in fact observe $k = r$ we are at least $100(1-\alpha)$ percent confident that $2nt^*/\chi^2_{\alpha,2r+2} < \theta$. Likewise, by considering $P(k \ge r|\theta)$, we may be $100(1-\alpha)$ percent confident that, given we observed $k = r$,

$2nt^*/\chi^2_{1-\alpha,2r} > \theta$. As a consequence, one sided and two sided $100(1-\alpha)$ percent confidence intervals for θ , under this sampling scheme, are given by

$$\left(\frac{2nt^*}{\chi^2_{\alpha,2r+2}}, \infty\right) \tag{4}$$

and

$$\left(\frac{2nt^*}{\chi^2_{\frac{\alpha}{2},2r+2}}, \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2},2r}}\right) . \tag{5}$$

This approach to confidence intervals for the Poisson parameter was originally discussed by Garwood [8]. Note that if $r = 0$, only the one-sided interval is used since we have no information on which to base an upper limit.

If the type I scheme proceeds without replacement, corresponding results are not easily obtained. In this case the distribution of the maximum likelihood estimator was shown by Bartholomew [2] to be a weighted sum of chi-square integrals. A closed form expression for confidence intervals from this is unlikely. Epstein used a non-parametric approach [4] to this problem by considering the number of failures observed in $(0,t^*)$, but ignoring the actual times of failure. We may improve our position considerably here if we choose to monitor total time on test rather than actual elapsed time. If at some time t , k failures have been observed, the total time on test is seen to be

$$\sum_{i=0}^{k} \tau_i + (n-k)t \ .$$

Instead of an actual time of truncation t^*, we preassign a total time truncation point t'. Again, Epstein [4] has considered this approach and many resultant properties. Under this scheme it can be shown that we are observing a Poisson process with parameter $\lambda = \frac{1}{\theta}$ for a length of <u>actual</u> <u>time</u> t'. Therefore, following the same reasoning that lead to (4) and (5), one-sided and two-sided $100(1-\alpha)$ percent confidence intervals for θ are

$$\left(\frac{2t'}{\chi^2_{\alpha,2r+2}}, \infty\right) \tag{6}$$

and

$$\left(\frac{2t'}{\chi^2_{\frac{\alpha}{2},2r+2}}, \frac{2t'}{\chi^2_{1-\frac{\alpha}{2},2r}}\right) . \tag{7}$$

With this scheme, the objective of terminating the test before an excessive amount of time has elapsed may still be achieved.

A more complex situation arises when types I and II are combined. In a combined scheme, the test may terminate either with r_0 failures or at time t^*, whichever occurs first. If we are testing $H_0\colon \theta = \theta_0$ against $H_1\colon \theta = \theta_1$ where $\theta_1 < \theta_0$, we would accept if $\min(\tau_{r_0}, t^*) = t^*$ and reject if $\min(\tau_{r_0}, t^*) = \tau_{r_0}$. It may be that r_0 and t^* are prechosen to satisfy α, β level requirements.

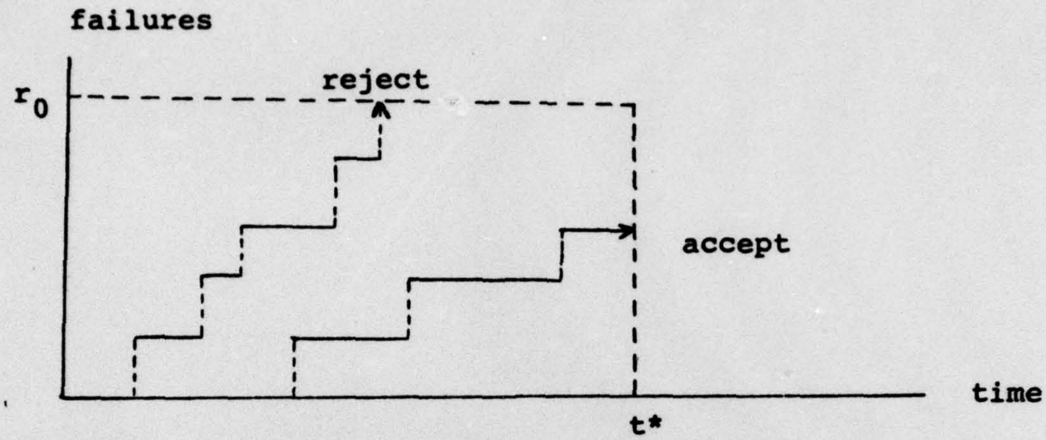


Figure 1. Possible sample paths in the combined test scheme.

Additional complexity in the stopping rule complicates the formulation of a confidence interval at the time the test terminates. The natural inclination here is to combine the confidence intervals from the types I and II. Epstein has done this and showed [4] that the following rule gives, at least, 100(1-α) percent one-sided confidence intervals <u>when</u> sampling with replacement:

$$\left(\frac{2nt^*}{\chi^2_{\alpha,2k+2}}, \infty\right) \quad \text{if} \quad \tau_{r_0} > t^*, \quad k = 0,1,\ldots,r_0-1 \tag{8}$$

$$\left(\frac{2n\tau_{r_0}}{\chi^2_{\alpha,2r_0}}, \infty\right) \quad \text{if} \quad \tau_{r_0} \le t^* .$$

If we sample without replacement we face the same problems as above when monitoring actual elapsed time. If, instead, we monitor total time on test and use total time truncation t', the following one sided $100(1-\alpha)$ percent confidence intervals for θ were also given by Epstein [4]:

$$\begin{aligned} &\left(\frac{2t'}{\chi^2_{\alpha,2k+2}}, \infty\right) \quad \text{if} \quad k = 0,1,\ldots,r_0-1 \\ &\left(\frac{2t_{r_0}}{\chi^2_{\alpha,2r_0}}, \infty\right) \quad \text{if} \quad k = r_0 . \end{aligned} \tag{9}$$

Here k is the number of failures observed when total accumulated test time is t', and

$$t_{r_0} = \sum_{i=1}^{r_0} \tau_i + (n-r_0)\tau_{r_0} = \text{total time on test at } r_0^{\text{th}} \text{ failure.}$$

Two sided intervals are considered in the next section.

III. Extension to Two Sided Intervals

IIIa. Testing with Replacement

It would seem that an extension from one sided to two sided intervals should now be simple and direct. In fact the proof is not a simple extension from the one sided case. Theorem 1 gives two sided intervals.

<u>Theorem 1</u>. Let τ be the lifetime of an item having distribution $f(\tau) = (1/\theta)e^{-\tau/\theta}$, $\tau \geq 0$. If n items from this distribution are placed on test with replacement, and if the test terminates at $\min(t^*, \tau_{r_0})$, then the following rule gives a two sided $100(1-\alpha)$ percent confidence interval for θ :

$$\begin{aligned}
&\left(\frac{2nt^*}{\chi^2_{\frac{\alpha}{2},2}},\ \infty\right) && \text{if } k=0 , \\
&\left(\frac{2nt^*}{\chi^2_{\frac{\alpha}{2},2k+2}},\ \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2},2k}}\right) && \text{if } k=1,2,\ldots,r_0-1 , \qquad (10)\\
&\left(\frac{2n\tau_{r_0}}{\chi^2_{\frac{\alpha}{2},2r_0}},\ \frac{2n\tau_{r_0}}{\chi^2_{1-\frac{\alpha}{2},2r_0}}\right) . && \text{if } \tau_{r_0} \leq t^* ,
\end{aligned}$$

Here k is the number of failures in $(0,t^*)$. Epstein proposed intervals nearly identical to these [4] with the exception that

he used $\chi^2_{\alpha,2}$ in the denominator when $k=0$. He was unable to provide a proof of his conjecture. To prove the intervals (1) are valid we need to show that, wherever θ be in $(0,\infty)$, the probability is at least $1-\alpha$ that θ is covered by one of the intervals (10). The proof is somewhat tedious and requires attention to figure 2. The endpoints of the intervals for $k=0,1,\ldots,r_0-1$ form a partition of the parameter space. The proof consists of establishing the required probability of $1-\alpha$ for each subset of the partition. Note first that the lower endpoints of the intervals when $k=0,1,2,\ldots,r_0-1$ are ordered among themselves, the smallest when $k=r_0-1$, the largest when $k=0$. This is a result of the fact that $\chi^2_{\alpha,k+1} > \chi^2_{\alpha,k}$ for any fixed α, and any integer k. Likewise the upper endpoints when $k=1,2,\ldots,r_0-1$ are also ordered, the smallest when $k=r_0-1$, the largest when $k=1$. The endpoints of the interval when $k=r_0$ are random since they are functions of τ_{r_0}, the time of the r_0th failure. However, $\tau_{r_0} < t^*$ implies this random lower endpoint is bounded above by the fixed lower endpoint of the interval corresponding to $k=r_0-1$, specifically $2n\tau_{r_0}/\chi^2_{\alpha/2,2r_0} \le 2nt^*/\chi^2_{\alpha/2,2r_0}$. Also this random upper endpoint is bounded above by the fixed upper endpoint when $k=r_0$, i.e., $2n\tau_{r_0}/\chi^2_{1-\alpha/2,2r_0} \le 2nt^*/\chi^2_{1-\alpha/2,2r_0}$. Note that if the smallest upper endpoint $(k=r_0-1)$ was always greater than the largest

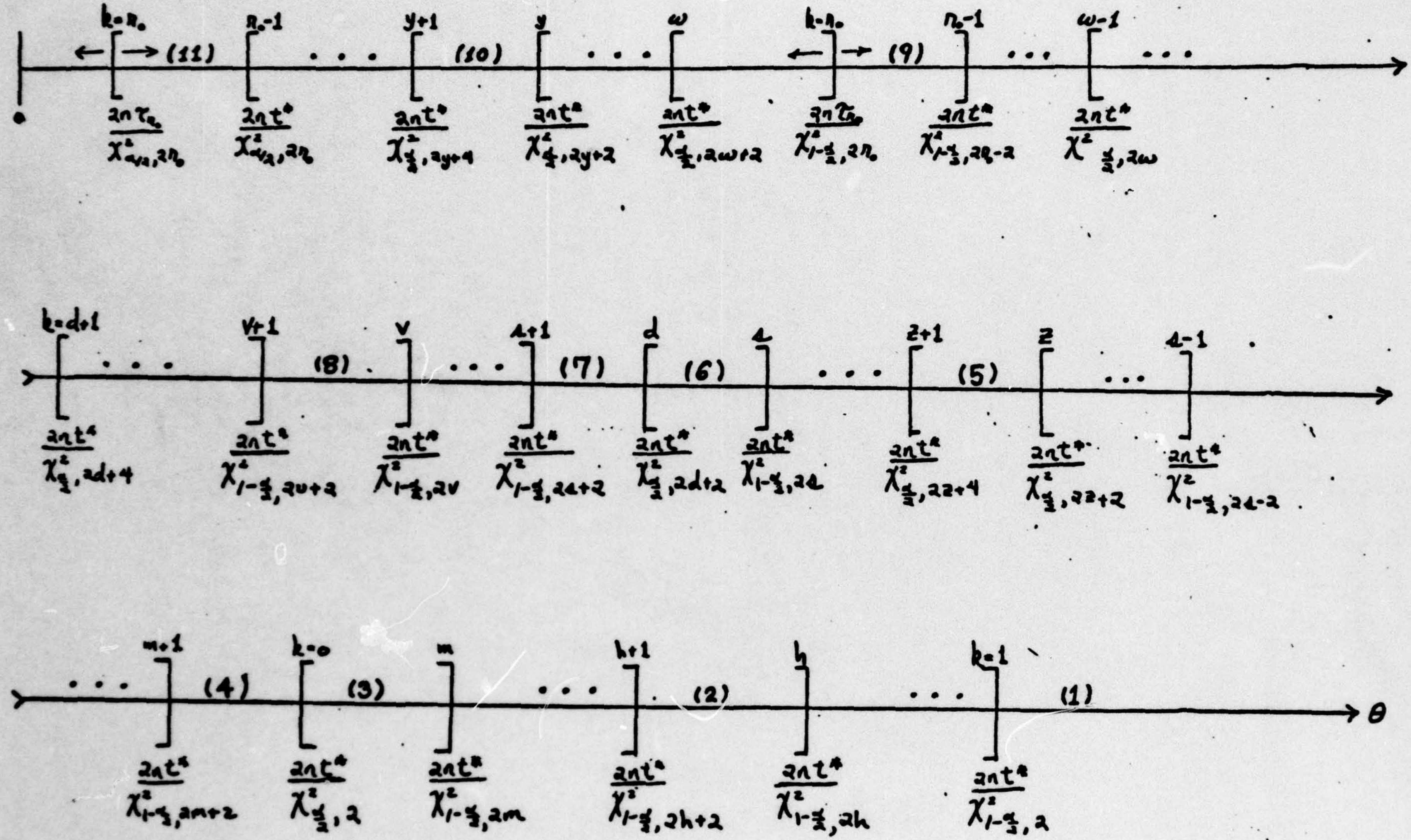


Figure 2. General alignment of interval endpoints in the replacement case

lower endpoint $(k=0)$, there would then be a convenient overall ordering. Unfortunately this is not necessarily the case. The partitioning of the parameter space may be complicated by what we shall refer to as a "cross-over". A cross-over occurs when an upper limit for one fixed interval crosses below the lower limit of another fixed interval. The first such cross-over will occur when $2nt^*/\chi^2_{1-\frac{\alpha}{2},2r_0-2}$ becomes less than $2nt^*/\chi^2_{\frac{\alpha}{2},2}$,

that is when $\chi^2_{1-\frac{\alpha}{2},2r_0-2} > \chi^2_{\frac{\alpha}{2},2}$. This can occur if α is large enough or, more likely, if r_0 is large enough for a fixed α level. Clearly a vast number of possible configurations exist where upper endpoints are intermixed with lower endpoints. One such configuration is illustrated in figure 3 below. In this illustration $\alpha = .10$ and $r_0 = 13$.

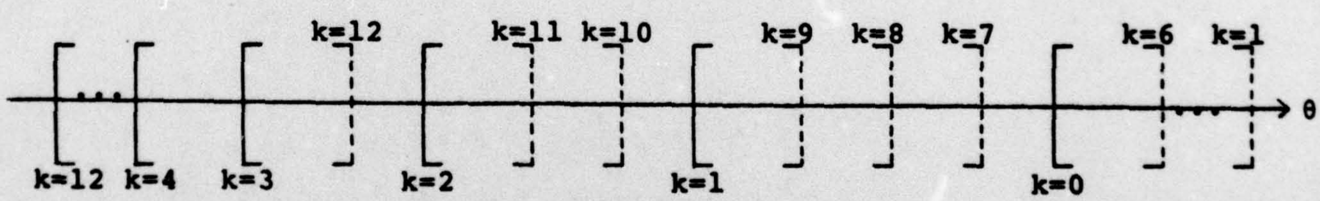


Figure 3. Example of interval endpoints intermixed, $\alpha = .10$, $r_0 = 13$.

All of these possible configurations may be reduced to the consideration of a relatively few cases as indicated in figure 2. Clearly we cannot consider, individually, all possible subsets of every possible configuration of endpoints in the partition. In each

conceivable case, however, the proof would utilize only a subset of the cases below.

We first prove two short lemmas that will be used in the general proof.

Lemma 1. If $x \geq 0$, then $f(x) = \sum_{k=0}^{n} e^{-x} x^k/k!$ is monotonically decreasing in x.

The proof is direct. We have $f'(x) = e^{-x} \sum_{k=0}^{n} (kx^{k-1} - x^k)/k!$, which is a telescoping sum. Hence $f'(x) = -e^{-x} x^n/n!$. Since $f'(x) \leq 0$ for all $x \geq 0$, the result follows.

Lemma 2. Let $\tau_1, \ldots, \tau_{r_0}$ be the first r_0 failure times out of n items on test, where each item has an exponential lifetime with parameter θ. That is $f(\tau_i) = (1/\theta)e^{-\tau_i/\theta}$, $\tau_i \geq 0$. Then $Y = 2n\tau_{r_0}/\theta$ has a chi-square distribution with $2r_0$ degrees of freedom.

To prove this we write $\tau_{r_0} = \tau_1 + (\tau_2 - \tau_1) + \ldots + (\tau_{r_0} - \tau_{r_0-1})$. Since we are observing a Poisson process with parameter $\lambda = n/\theta$ it follows that τ_{r_0} is the sum of r_0 independent, identically distributed random variables with common exponential distribution $f(x) = \frac{n}{\theta} e^{-nx/\theta}$, $x \geq 0$. Consequently τ_{r_0} has a gamma distribution with parameters $\alpha = r_0$, $\beta = \theta/n$. If we let $Y = 2n\tau_{r_0}/\theta$, the distribution of Y is easily found to be $\chi^2_{2r_0}$ from the transformation on τ_{r_0}.

Proof of Theorem 1. Case (i) corresponds to region i in figure 2.

Let I represent the family of intervals (10), and k be the number of failures observed in $(0,t^*)$.

Case (1): $\theta \geq \dfrac{2nt^*}{\chi^2_{1-\frac{\alpha}{2},2}}$.

Under this restriction on θ , $P(\theta \in I) = P(\text{no failures before } t^*)$ $= e^{-\frac{nt^*}{\theta}}$, since we are observing a Poisson process with parameter $\frac{n}{\theta}$. By the case (1) assumption $nt^*/\theta \leq \chi^2_{1-\frac{\alpha}{2},2}/2$ which gives

$$e^{-\frac{nt^*}{\theta}} \geq e^{-\chi^2_{1-\frac{\alpha}{2},2}/2} .$$

Now we can make use of the well known relationship

$$P(\chi^2_{2r+2} > y) = \sum_{k=0}^{r} \frac{e^{-y/2}(y/2)^k}{k!} . \tag{11}$$

Thus, with $r = 0$, $P(\chi^2_2 > \chi^2_{1-\frac{\alpha}{2},2}) = e^{-\chi^2_{1-\frac{\alpha}{2},2}/2} = 1 - \frac{\alpha}{2}$. So we have $P(\theta \in I) \geq 1 - \frac{\alpha}{2} > 1-\alpha$ which was to be proved for case (1).

Case (2): $\dfrac{2nt^*}{\chi^2_{1-\frac{\alpha}{2},2h+2}} \leq \theta < \dfrac{2nt^*}{\chi^2_{1-\frac{\alpha}{2},2h}}$.

$$P(\theta \in I) = P\left[\bigcup_{r=0}^{h} (r \text{ failures before } t^*)\right] = \sum_{r=0}^{h} P(r \text{ failures in } t^*)$$

$$= \sum_{r=0}^{h} \frac{e^{-\frac{nt^*}{\theta}}(\frac{nt^*}{\theta})^r}{r!} .$$

By case (2) assumption, $nt^*/\theta \le \chi^2_{1-\frac{\alpha}{2},2h+2}/2$ which implies

$$\sum_{r=0}^{h} \frac{e^{-\frac{nt^*}{\theta}}\left(\frac{nt^*}{\theta}\right)^r}{r!} \ge \sum_{r=0}^{h} \frac{e^{-\chi^2_{1-\frac{\alpha}{2},2h+2}/2}\left(\chi^2_{1-\frac{\alpha}{2},2h+2}/2\right)^r}{r!} = 1 - \frac{\alpha}{2} .$$

The inequality follows from Lemma 1 and the equality from (11). Hence, again, $P(\theta \in I) > 1 - \alpha$.

Case (3): $\dfrac{2nt^*}{\chi^2_{\frac{\alpha}{2},2}} \le \theta < \dfrac{2nt^*}{\chi^2_{1-\frac{\alpha}{2},2m}}$.

This is included in the more general case (6) with $d = 0$ and $s = m$.

Case (4): $\dfrac{2nt^*}{\chi^2_{1-\frac{\alpha}{2},2m+2}} \le \theta < \dfrac{2nt^*}{\chi^2_{\frac{\alpha}{2},2}}$.

Again, this is included in case (7) with $d = 0$, $s = m$.

Case (5): $\dfrac{2nt^*}{\chi^2_{\frac{\alpha}{2},2z+4}} \le \theta < \dfrac{2nt^*}{\chi^2_{\frac{\alpha}{2},2z+2}}$.

Note here that we necessarily have $s \ge d+1$. Otherwise, if $s = d$, we would have $\dfrac{2nt^*}{\chi^2_{1-\frac{\alpha}{2},2d}} < \dfrac{2nt^*}{\chi^2_{\frac{\alpha}{2},2d}}$ or $\chi^2_{1-\frac{\alpha}{2},2d} > \chi^2_{\frac{\alpha}{2},2d}$.

which is clearly not possible. Now within case (5),

$$P(\theta \in I) = P\left[\bigcup_{r=z+1}^{s-1} (\text{r failures in } t^*)\right] = \sum_{r=z+1}^{s-1} P(\text{r failures in } t^*) =$$

$$= \sum_{r=z+1}^{s-1} e^{-\frac{nt^*}{\theta}} (\tfrac{nt^*}{\theta})^r/r! = \sum_{r=0}^{s-1} e^{-\frac{nt^*}{\theta}} (\tfrac{nt^*}{\theta})^r/r! - \sum_{r=0}^{z} e^{-\frac{nt^*}{\theta}} (\tfrac{nt^*}{\theta})^r/r! \ .$$

By case (5) assumption $\theta > \dfrac{2nt^*}{\chi^2_{1-\frac{\alpha}{2},2s}}$ and so $\dfrac{nt^*}{\theta} < \dfrac{\chi^2_{1-\frac{\alpha}{2},2s}}{2}$.

Consequently the first sum is $> \sum_{r=0}^{s-1} e^{-\chi^2_{1-\frac{\alpha}{2},2s/2}} (\chi^2_{1-\frac{\alpha}{2},2s}/2)^r/r! = 1 - \frac{\alpha}{2}$,

and $P(\theta \in I) > 1 - \frac{\alpha}{2} - \sum_{r=0}^{z} e^{-\frac{nt^*}{\theta}} (\frac{nt^*}{\theta})^r/r!$. But

$nt^*/\theta > \chi^2_{\frac{\alpha}{2},2z+2}/2$ and so

$$\sum_{r=0}^{z} e^{-\frac{nt^*}{\theta}} (\tfrac{nt^*}{\theta})^r/r! < \sum_{r=0}^{z} e^{-\chi^2_{\alpha/2,2z+2}/2} (\chi^2_{\frac{\alpha}{2},2z+2}/2)^r/r! = \frac{\alpha}{2} \ .$$

Finally $P(\theta \in I) > 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha$.

Case (6): $\dfrac{2nt^*}{\chi^2_{1-\frac{\alpha}{2},2s+2}} < \dfrac{2nt^*}{\chi^2_{\frac{\alpha}{2},2d+2}} \le \theta < \dfrac{2nt^*}{\chi^2_{1-\frac{\alpha}{2},2s}} < \dfrac{2nt^*}{\chi^2_{\frac{\alpha}{2},2d}}$.

$$P(\theta \in I) = P\left[\bigcup_{r=d}^{s} (r \text{ failures in } t^*)\right] = \sum_{r=d}^{s} e^{-\frac{nt^*}{\theta}} (\tfrac{nt^*}{\theta})^r/r! =$$

$$\sum_{r=0}^{s} e^{-\frac{nt^*}{\theta}} (\tfrac{nt^*}{\theta})^r/r! - \sum_{r=0}^{d-1} e^{-\frac{nt^*}{\theta}} (\tfrac{nt^*}{\theta})^r/r! >$$

$$> \sum_{r=0}^{s} e^{-\chi^2_{1-\frac{\alpha}{2},2s+2}/2} (\chi^2_{1-\frac{\alpha}{2},2s+2}/2)^r/r! - \sum_{r=0}^{d-1} e^{-\chi^2_{\alpha/2,2d}/2} (\chi^2_{\frac{\alpha}{2},2d}/2)^r/r! \, .$$

The last inequality follows from case (6) assumption which implies

$$\frac{nt^*}{\theta} < \frac{\chi^2_{1-\frac{\alpha}{2},2s+2}}{2} \quad \text{and} \quad \frac{nt^*}{\theta} > \frac{\chi^2_{\frac{\alpha}{2},2d}}{2} \, .$$

Consequently $P(\theta \in I) > 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha$.

Case (7): $\dfrac{2nt^*}{\chi^2_{1-\frac{\alpha}{2},2s+2}} \le \theta < \dfrac{2nt^*}{\chi^2_{\frac{\alpha}{2},2d+2}}$.

$$P(\theta \in I) = P\left[\bigcup_{r=d+1}^{s} (r \text{ failures in time } t^*)\right] =$$

$$= \sum_{r=0}^{s} e^{-\frac{nt^*}{\theta}} (\tfrac{nt^*}{\theta})^r/r! - \sum_{r=0}^{d} e^{-\frac{nt^*}{\theta}} (\tfrac{nt^*}{\theta})^r/r! >$$

$$\sum_{r=0}^{s} e^{-\chi^2_{1-\frac{\alpha}{2},2s+2}/2} (\chi^2_{1-\frac{\alpha}{2},2s+s}/2)^r/r!$$

$$- \sum_{r=0}^{d} e^{-\chi^2_{\frac{\alpha}{2},2d+2}/2} (\chi^2_{\frac{\alpha}{2},2d+2}/2)^r/r! \quad .$$

Again the inequality follows from the restriction on θ which implies $nt^*/\theta \le \chi^2_{1-\frac{\alpha}{2},2s+2}/2$ and $nt^*/\theta > \chi^2_{\alpha/2,2d+2}/2$. So again $P(\theta \in I)$ $1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha$. It is noteworthy here to consider this case when $d = 0$, $s = m$, which is merely case (4) as previously cited. The proof here relies on the fact that $\theta < 2nt^*/\chi^2_{\frac{\alpha}{2},2}$. But in Epstein's conjectured intervals, this partition point would be $2nt^*/\chi^2_{\alpha,2}$, which gives $P(\theta \in I)$ $1 - \frac{\alpha}{2} - \alpha = 1 - \frac{3\alpha}{2}$. Problems arise if $2nt^*/\chi^2_{\alpha/2,2} < \theta < 2nt^*/\chi^2_{\alpha,2}$. For example, it can be shown that for the simple case with $r_0 = 2$, with θ in this small interval, $P(\theta \in I) < 1 - \alpha$.

Case (8): $\dfrac{2nt^*}{\chi^2_{1-\frac{\alpha}{2},2v+2}} \le \theta < \dfrac{2nt^*}{\chi^2_{1-\frac{\alpha}{2},2v}}$.

$P(\theta \in I) = \sum_{r=d+1}^{v} P(r \text{ failures before time } t^*) =$

$$\sum_{r=0}^{v} e^{-\frac{nt^*}{\theta}} (\frac{nt^*}{\theta})^r/r! - \sum_{r=0}^{d} e^{-\frac{nt^*}{\theta}} (\frac{nt^*}{\theta})^r/r! >$$

$$> \sum_{r=0}^{v} e^{-\chi^2_{1-\frac{\alpha}{2},2v+2}/2} (\chi^2_{1-\frac{\alpha}{2},2v+2}/2)^r/r! - \sum_{r=0}^{d} e^{-\chi^2_{\alpha/2,2d+2}/2} (\chi^2_{\frac{\alpha}{2},2d+2}/2)^r/r! .$$

Again this follows from $nt^*/\theta \le \chi^2_{1-\frac{\alpha}{2},2v+2}$ and $nt^*/\theta > \chi^2_{\alpha/2,2d+2}$.

So, for case (8), $P(\theta \in I) > 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha$.

Case (9): $\dfrac{2nt^*}{\chi^2_{\frac{\alpha}{2},2w+2}} \le \theta < \dfrac{2nt^*}{\chi^2_{1-\frac{\alpha}{2},2r_0-2}}$.

Note that now we must include the possibility of coverage by the random interval corresponding to $k = r_0$.

$$P(\theta \in I) = \sum_{r=w}^{r_0-1} P(r \text{ failures before } t^*) + P(r_0 \text{ fail and } \theta \le \frac{2n\tau_{r_0}}{\chi^2_{1-\frac{\alpha}{2},2r_0}})$$

$$= \sum_{r=w}^{r_0-1} e^{-\frac{nt^*}{\theta}} (\frac{nt^*}{\theta})^r/r! + P(\frac{\theta\chi^2_{1-\frac{\alpha}{2},2r_0}}{2n} \le \tau_{r_0} \le t^*) .$$

It follows from Lemma 2 that we can write this second term as

$$P(\frac{\theta\chi^2_{1-\frac{\alpha}{2},2r_0}}{2n} \le \tau_{r_0} \le t^*) = P(\chi^2_{1-\frac{\alpha}{2},2r_0} \le Y \le \frac{2nt^*}{\theta})$$

$$= F_y(2nt^*/\theta) - \frac{\alpha}{2} = 1 - \frac{\alpha}{2} - P(\chi^2_{2r_0} > \frac{2nt^*}{\theta}) \doteq 1 - \frac{\alpha}{2} - \sum_{r=0}^{r_0-1} e^{-\frac{nt^*}{\theta}} (\frac{nt^*}{\theta})^r/r! .$$

$$P(\theta \in I) = \sum_{r=w}^{r_0-1} e^{-\frac{nt^*}{\theta}} (\frac{nt^*}{\theta})^r/r! + 1 - \frac{\alpha}{2} - \sum_{r=0}^{r_0-1} e^{-\frac{nt^*}{\theta}} (\frac{nt^*}{\theta})^r/r!$$

$$= 1 - \frac{\alpha}{2} - \sum_{r=0}^{w-1} e^{-\frac{nt^*}{\theta}} (\frac{nt^*}{\theta})^r/r! \ .$$

But, by assumption, $\theta < 2nt^*/\chi^2_{\alpha/2,2w}$ and so $nt^*/\theta > \chi^2_{\alpha/2,2w/2}$

which then implies $P(\theta \in I) > 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha$.

Case (10): $\dfrac{2nt^*}{\chi^2_{\alpha/2,2y+4}} \le \theta < \dfrac{2nt^*}{\chi^2_{\alpha/2,2y+2}}$.

$$P(\theta \in I) = \sum_{r=y+1}^{r_0-1} P(r \text{ fail in } t^*) + P(r_0 \text{ fail and } \theta \le \frac{2n\tau_{r_0}}{\chi^2_{1-\frac{\alpha}{2},2r_0}})$$

$$= \sum_{r=y+1}^{r_0-1} e^{-\frac{nt^*}{\theta}} (\frac{nt^*}{\theta})^r/r! + P(\chi^2_{1-\frac{\alpha}{2},2r_0} \le Y \le \frac{2nt^*}{\theta}) \quad \text{as in case (9).}$$

By the same reasoning as in case (9),

$P(\theta \in I) = 1 - \frac{\alpha}{2} - \sum_{r=0}^{y} e^{-\frac{nt^*}{\theta}} (\frac{nt^*}{\theta})^r/r!$ and since,

$nt^*/\theta > \chi^2_{\frac{\alpha}{2},2y+2}/2$, $P(\theta \in I) > 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha$.

Case (ii): $\theta < \frac{2nt^*}{\chi^2_{\frac{\alpha}{2},2r_0}}$.

$$P(\theta \in I) = P\left[\frac{2n\tau_{r_0}}{\chi^2_{\frac{\alpha}{2},2r_0}} \le \theta \le \frac{2n\tau_{r_0}}{\chi^2_{1-\frac{\alpha}{2},2r_0}}\right]$$

$$= P\left[\frac{\theta\chi^2_{1-\alpha/2,2r_0}}{2n} \le \tau_{r_0} \le \frac{\theta\chi^2_{\alpha/2,2r_0}}{2n}\right]$$

$$= P(\chi^2_{1-\frac{\alpha}{2},2r_0} \le \chi^2_{2r_0} \le \chi^2_{\frac{\alpha}{2},2r_0}) = 1-\alpha .$$

These cases cover any possible subset of every partition. A formal proof of this claim is found in Appendix 1. The proof of Theorem 1 is complete.

IIIb. Expected Interval Length

It is of some interest now to derive an expression for the expected length of the confidence interval under this scheme. There is a positive probability of observing no failures, in which case the interval length is infinite. To avoid this situation we will condition on one or more failures. Denote the length of the interval by L and consider the following two possibilities:

If $\tau_{r_0} > t^*$ then $L = 2nt^*\left[\frac{1}{\chi^2_{1-\frac{\alpha}{2},2k}} - \frac{1}{\chi^2_{\frac{\alpha}{2},2k+2}}\right] = \ell_k$, say.

Here k is the number of failures observed with $1 \le k \le r_0 - 1$.

If $\tau_{r_0} \le t^*$ then $L = \left[\frac{2n}{\chi^2_{1-\frac{\alpha}{2},2r_0}} - \frac{2n}{\chi^2_{\frac{\alpha}{2},2r_0}}\right]\tau_{r_0} = c\tau_{r_0}$, say.

Note that in the first case the length is discrete while in the second it is a continuous random variable. Consequently the unconditional distribution of L is of the mixed type with jumps at ℓ_k, $k = 1,2,\ldots,r_0-1$. So,

$$F(\ell) = \int_0^{\ell/c} f(t)dt + \sum_{k\in A} P(k \text{ failures in } t^*) \quad \text{if} \quad \ell < ct^*$$

$$= \int_0^{t^*} f(t)dt + \sum_{k\in A} P(k \text{ failures in } t^*) \quad \text{if} \quad \ell \ge ct^* ,$$

where f is the density function of τ_{r_0} and $A = \{k: \ell_k \le \ell\}$.

From Lemma 2 we have that $\tau_{r_0} \sim$ gamma $(\alpha = r_0, \beta = \theta/n)$, and $P(k \text{ failures}) = e^{-\frac{nt^*}{\theta}}(\frac{nt^*}{\theta})^k/k!$. We want $E(L|L < \infty)$ which is $E(L|k \ne 0)$ since $L = \infty$ if and only if $k = 0$. We need now the conditional distribution of L . We can write $P(L \le \ell | L < \infty) = P(L \le \ell \cap L < \infty)/P(k \ne 0)$. Now if ℓ is finite,

$(L \le \ell) \subset (L < \infty)$ which then implies $P(L \le \ell \cap L < \infty) = P(L \le \ell)$.

On the other hand, if ℓ is infinite, $(L < \infty) \subset (L \le \ell)$ and $P(L \le \ell \cap L < \infty) = P(L < \infty)$. Thus

$$F(\ell | L < \infty) = \frac{F(\ell)}{P(k \ne 0)} \quad \text{if } \ell \text{ is finite}$$

$$= 1 \quad \text{if } \ell \text{ is infinite.}$$

Consequently, $E(L|k\ne 0) = \int_0^\infty \ell \, dF(\ell|k\ne 0) = \frac{1}{P(k\ne 0)} \int_0^\infty \ell \, dF(\ell)$,

where $P(k\ne 0) = 1 - e^{-\frac{nt^*}{\theta}}$. Thus,

$$E(L|k\ne 0) = \frac{1}{P(k\ne 0)}\left[\sum_{k=1}^{r_0-1} \ell_k P(k \text{ fail in } t^*) + \int_0^{ct^*} \ell g(\ell) d\ell\right]$$

$$= \frac{1}{P(k\ne 0)}\left[\sum_{k=1}^{r_0-1} \ell_k P(k \text{ fail in } t^*) + \int_0^{t^*} c\tau_{r_0} f(\tau_{r_0}) d\tau_{r_0}\right] .$$

To evaluate the integral:

$$c \int_0^{t^*} y f(y) dy = c \int_0^{t^*} \frac{y^{r_0} \cdot e^{-\frac{ny}{\theta}}}{\Gamma(r_0)\left(\frac{\theta}{n}\right)^{r_0}} dy$$

$$= \frac{c}{\Gamma(r_0)\left(\frac{\theta}{n}\right)^{r_0}} \left[e^{-\frac{ny}{\theta}} \sum_{k=0}^{r_0} \cdot \frac{(-1)^k r_0! \, y^{r_0-k}}{(r_0-k)!\left(-\frac{\theta}{n}\right)^{k+1}} \right]_{y=0}^{t^*}$$

$$= \frac{c\, r_0\, \theta}{n} \left[1 - \sum_{k=0}^{r_0} \frac{e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^k}{k!} \right] .$$

Thus,

$$E(L|k\neq 0) = \frac{1}{1-e^{-\frac{nt^*}{\theta}}} \left[\sum_{k=1}^{r_0-1} \ell_k \frac{e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^k}{k!} + \frac{\theta\, c\, r_0}{n} \sum_{k=r_0+1}^{\infty} \frac{e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^k}{k!} \right] \tag{12}$$

where $\ell_k = 2nt^* \left[\frac{1}{\chi^2_{1-\frac{\alpha}{2}, 2k}} - \frac{1}{\chi^2_{\frac{\alpha}{2}, 2k+2}} \right]$

and $c = 2n \left[\frac{1}{\chi^2_{1-\frac{\alpha}{2}, 2r_0}} - \frac{1}{\chi^2_{\frac{\alpha}{2}, 2r_0}} \right]$.

Several questions may come to mind concerning the expression for $E(L|k\neq 0)$. For instance, if we take the limit as $r_0 \to \infty$ in (12), do we obtain the $E(L|k\neq 0)$ for the type I censoring scheme? If we take the limit as $t^* \to \infty$, do we obtain the $E(L)$ for the type II censoring scheme? It can be shown quite easily that the answer to both questions is yes. In the preceding derivations we have placed no restrictions on t^* and r_0 other than to say they are censoring, or truncation, values. In practice the values

of t^* and r_0 are determined by the test procedure which provides the data for estimation. MIL-STD-781 is concerned with testing H_0: $\theta = \theta_0$ against H_1: $\theta = \theta_1$ where $\theta_1 < \theta_0$. If a test combining type I and type II censoring with $P(\text{reject } H_0 | \theta_1) = \alpha$ is desired, Epstein [5] has shown that $t^* = \theta_0 \chi^2_{1-\alpha, 2r_0}/2n$. If in addition we require $P(\text{accept } H_1 | \theta_1) \le \beta$, Epstein has shown this may be achieved if we choose r_0 to be the smallest integer such that

$$\frac{\theta_0}{\theta_1} \ge \frac{\chi^2_{\beta, 2r_0}}{\chi^2_{1-\alpha, 2r_0}} \quad .$$

Then we may decrease β if we increase r_0. It seems reasonable that an increase in r_0, or a decrease in β, would result in a corresponding decrease in $E(L|k \neq 0)$. An attempt at a direct analytic proof using (12) was not successful. We then programmed (12) and generated $E(L|k \neq 0)$ for various combinations of θ, θ_0 and α. The value of r_0 was increased for each combination. The results, somewhat surprisingly, show that for some combinations of θ, θ_0 and α, the expected length <u>increases</u> initially before it does decrease with increasing r_0. Partial results of this computer work are given in Table 1. We have, thus far, been unable to provide a satisfactory explanation of this behavior.

Table I. Expected Lengths for Increasing Truncation Values, R

$\theta = 1.00 \qquad \theta_0 = 3.00$

$\alpha = 0.05$		$\alpha = 0.10$	
R = 2	E(L) = 25.7057	R = 2	E(L) = 14.6718
R = 3	E(L) = 26.1465	R = 3	E(L) = 11.1729
R = 4	E(L) = 15.0962	R = 4	E(L) = 5.6256
R = 5	E(L) = 6.9176	R = 5	E(L) = 2.8950
R = 6	E(L) = 3.4234	R = 6	E(L) = 1.9311
R = 7	E(L) = 2.2408	R = 7	E(L) = 1.5818
R = 8	E(L) = 1.8238	R = 8	E(L) = 1.4094
R = 9	E(L) = 1.6299	R = 9	E(L) = 1.2954
R = 10	E(L) = 1.5034	R = 10	E(L) = 1.2071
R = 11	E(L) = 1.4068	R = 11	E(L) = 1.1351
R = 12	E(L) = 1.3266	R = 12	E(L) = 1.0745
R = 13	E(L) = 1.2590	R = 13	E(L) = 1.0223
R = 14	E(L) = 1.2001	R = 14	E(L) = 0.9771
R = 15	E(L) = 1.1490	R = 15	E(L) = 0.9374

$\theta = 2.00 \qquad \theta_0 = 3.00$

$\alpha = 0.05$		$\alpha = 0.10$	
R = 2	E(L) = 33.8920	R = 2	E(L) = 22.2509
R = 3	E(L) = 53.1913	R = 3	E(L) = 28.8162
R = 4	E(L) = 54.6214	R = 4	E(L) = 25.6829
R = 5	E(L) = 44.3260	R = 5	E(L) = 18.7492
R = 6	E(L) = 30.9006	R = 6	E(L) = 12.2808
R = 7	E(L) = 19.6450	R = 7	E(L) = 7.7827
R = 8	E(L) = 12.0050	R = 8	E(L) = 5.1069
R = 9	E(L) = 7.4825	R = 9	E(L) = 3.6492
R = 10	E(L) = 5.0097	R = 10	E(L) = 2.8815
R = 11	E(L) = 3.7223	R = 11	E(L) = 2.4700
R = 12	E(L) = 3.0517	R = 12	E(L) = 2.2326
R = 13	E(L) = 2.6882	R = 13	E(L) = 2.0790
R = 14	E(L) = 2.4717	R = 14	E(L) = 1.9683
R = 15	E(L) = 2.3277	R = 15	E(L) = 1.8805

$\theta = 4.00 \qquad \theta_0 = 3.00$

$\alpha = 0.05$		$\alpha = 0.10$	
R = 2	E(L) = 38.6661	R = 2	E(L) = 27.0906
R = 3	E(L) = 73.4801	R = 3	E(L) = 44.5701
R = 4	E(L) = 97.9833	R = 4	E(L) = 54.0796
R = 5	E(L) =109.0797	R = 5	E(L) = 56.0089
R = 6	E(L) =108.3810	R = 6	E(L) = 52.4240
R = 7	E(L) = 99.3520	R = 7	E(L) = 45.7010
R = 8	E(L) = 85.6724	R = 8	E(L) = 37.8212
R = 9	E(L) = 70.4874	R = 9	E(L) = 30.1389
R = 10	E(L) = 55.8853	R = 10	E(L) = 23.4156
R = 11	E(L) = 43.1227	R = 11	E(L) = 17.9371
R = 12	E(L) = 32.6525	R = 12	E(L) = 13.6968
R = 13	E(L) = 24.4603	R = 13	E(L) = 10.5338
R = 14	E(L) = 18.3067	R = 14	E(L) = 8.2472
R = 15	E(L) = 13.8143	R = 15	E(L) = 6.6256

IIIc. Testing without Replacement

The second testing scheme we are considering terminates at $\min(\tau_{r_0}, t')$, where failed items are not replaced. Recall that the truncation value t' now refers to the total accumulated test time. At the time of decision the following rule gives a $100(1-\alpha)$ percent confidence interval for θ :

$$\left(\frac{2t'}{\chi^2_{\frac{\alpha}{2},2}}, \infty\right) \qquad \text{if } k=0 ,$$

$$\left(\frac{2t'}{\chi^2_{\frac{\alpha}{2},2k+2}}, \frac{2t'}{\chi^2_{1-\frac{\alpha}{2},2k}}\right) \qquad \text{if } k=1,2,\ldots,r_0-1, \tag{13}$$

$$\left(\frac{2T_{r_0}}{\chi^2_{\frac{\alpha}{2},2r_0}}, \frac{2T_{r_0}}{\chi^2_{1-\frac{\alpha}{2},2r_0}}\right) \qquad \text{if } k=r_0 .$$

Here k is the number of failures observed when the total time on test reaches t', and T_{r_0} is the total time on test at the time of the r_0th failure. This rule is nearly identical to that conjectured by Epstein [4], with again the only change being the use of $\alpha/2$ rather than α when $k=0$. The proof of (13) becomes simple when we realize that this scheme may be viewed as

a Poisson process with parameter $1/\theta$ which we observe for <u>actual time</u> t' . To see this, let T_k be the total time on test when the k^{th} item fails. Then

$$T_k = \sum_{i=1}^{k} \tau_i + (n-k)\tau_k = \sum_{i=1}^{k} (n-i+1)(\tau_i - \tau_{i-1}) = \sum_{i=1}^{k} U_i$$

where we have previously noted the U_i are independent and identically distributed as exponential $(\frac{1}{\theta})$. It follows that $T_k \sim$ gamma $(\alpha = k,\ \beta = \theta)$. We may write

P(k failures in total time t') = $P(T_{k+1} > t') - P(T_k > t')$. From direct integration we find that

$$P(T_{k+1} > t') = \sum_{r=0}^{k} e^{-\frac{t'}{\theta}} (\frac{t'}{\theta})^k / k!$$

$$P(T_k > t') = \sum_{r=0}^{k-1} e^{-\frac{t'}{\theta}} (\frac{t'}{\theta})^k / k!$$

and P(k failures) = $e^{-\frac{t'}{\theta}} (\frac{t'}{\theta})^k / k!$. Hence, to prove (13) we need only duplicate the proof of (10) substituting t' for t^* and $1/\theta$ for n/θ . The derivation of $E(L|k\neq 0)$ for this scheme follows in the same manner. The result is

$$E(L|k\neq 0) = \frac{1}{1-e^{-t'/\theta}} \left[\sum_{k=1}^{r_0-1} \ell_k' \frac{e^{-\frac{t'}{\theta}} (\frac{t'}{\theta})^k}{k!} + c' r_0 \theta \sum_{k=r_0+1}^{\infty} \frac{e^{-\frac{t'}{\theta}} (\frac{t'}{\theta})^k}{k!} \right]$$

where $\ell_k = 2t'\left(\frac{1}{\chi^2_{1-\frac{\alpha}{2},\,2k}} - \frac{1}{\chi^2_{\frac{\alpha}{2},\,2k+2}}\right) \qquad k = 1,2,\ldots,r_0-1$,

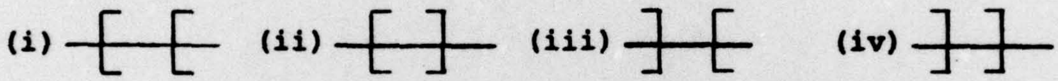
and $c' = 2\left(\frac{1}{\chi^2_{1-\frac{\alpha}{2},\,2r_0}} - \frac{1}{\chi^2_{\frac{\alpha}{2},\,2r_0}}\right)$.

Other results from the previous scheme carry over similarly.

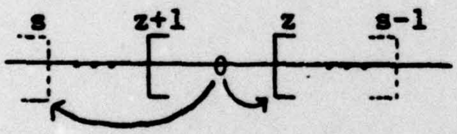
Appendix I

We now give a proof that the cases considered in Theorem 1 do in fact cover every subset of the possible partitions of the parameter space determined by the fixed interval endpoints.

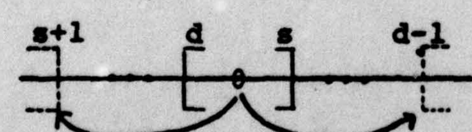
With the exception of cases (1) and (11) (regions 1 and 11 in figure 2), each possible subset in the partition will be bounded above and below by a left or right interval endpoint. Consequently there are only four possible subsets:



We consider each of these and show how each is covered in Theorem 1.

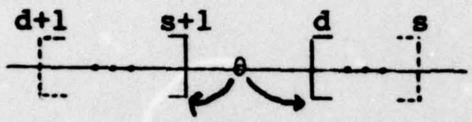
(i) 

In the most general situation this is case (5) of Theorem 1. The arrows indicate the bounds on θ which are necessary for the proof of case (5). It is possible that there is no right endpoint below the left endpoint determined by $k = z+1$. In this event we have $s-1 = r_0-1$ and we are then in case (10) of Theorem 1.

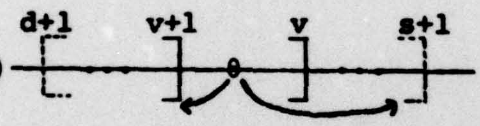
(ii) 

In general this is case (6) of Theorem 1. Again the arrows indicate the boundary points necessary for its proof. If there is no right endpoint corresponding to $k = s+1$, it must be that

$s = r_0 - 1$ and we have case (9) with $d = w$. If there is no left endpoint indicated by $d-1$ we have the situation where $d=0$ and $s=m$ which is case (3). If neither of the points indicated by $s+1$ and $d-1$ are found, we must have the situation where no crossover occurs. Then we are in case (9) with $w=0$.

(iii) 

This is case (7) in general and its proof is dependent only on the actual boundaries of the subset. No other considerations are necessary.

(iv) 

In general this is case (8) of Theorem 1. If the endpoint indicated by $k = s+1$ is not present, we must be in case (2) with $d+1 = 0$ and $v=h$.

By adding regions (1) and (11) in figure 2, we see that Theorem 1 covers all possible subsets. Cases (3) and (4) are included only for clarity in figure 2 and may be omitted, since they are special cases of (6) and (7).

References

[1] Bartholomew, D. J. (1963). "The Sampling Distribution of an Estimate Arising in Life Testing", Technometrics, 5, pp. 361-374.

[2] Epstein, B. (1960). Statistical Techniques in Life Testing, Distributed by the U. S. Department of Commerce, Office of Technical Services, Washington 25, D.C.

[3] Epstein, B. (1954). "Truncated Life Tests in the Exponential Case", Annals of Mathematical Statistics, 25, pp. 555-564.

[4] Garwood, F. (1936). "Fiducial Limits for the Poisson Distribution", Biometrika, 28, pp. 437-441.

[5] MIL-STD-781C, Military Standard Reliability Qualification and Production Acceptance Tests: Exponential Distribution, Draft, August 30, 1976.

[6] Wald, A. (1947). Sequential Analysis, John Wiley & Sons, New York.